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LETTER TO THE EDITOR

Scaling behaviour at the onset of mutual entrainment in a population of interacting oscillators

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Abstract. Scaling behaviour of an order parameter and its fluctuations is numerically investigated at the onset of macroscopic mutual entrainment in a population of interacting self-oscillators. In particular, evidence is presented for the power law divergence of the fluctuations with exponents near $\frac{1}{8}$. Finite-size scaling forms are also proposed and verified.

Populations of interacting self-oscillators with distributed natural frequencies can exhibit remarkable transitions from a disordered state to an ordered one as the strength of interactions is varied [1, 2] (see also references in [3, 4]), namely, as the strength exceeds a certain threshold, a macroscopic number of elements begin to be mutually entrained with a common frequency. A number is called macroscopic if its magnitude is comparable to the population size N which is assumed to be very large. Such a phenomenon is quite analogous to second-order phase transitions in equilibrium systems [5] so that it provides us with a new and interesting subject in the field of critical phenomena. The onset of mutual entrainment (ME) in large assemblies of oscillators may also be of significance in a variety of scientific disciplines such as biology [1] as well as in physics.

Although phase transitions in populations of oscillators have been studied previously using continuous-time models, i.e. differential equations, the author has recently proposed a class of discrete-time models and has investigated the onset of ME in a particular example as follows [3, 4]:

$$\theta_{n+1}^{(j)} = \theta_n^{(j)} + \Omega_j + \frac{\varepsilon}{2\pi N} \sum_{i=1}^{N} \sin[2\pi(\theta_n^{(i)} - \theta_n^{(j)})] \tag{1}$$

for $1 \leqslant j \leqslant N$ which is a discrete-time version of a model used by Kuramoto [2, 6–8], i.e.

$$\frac{\mathrm{d}\theta^{(j)}}{\mathrm{d}t} = \Omega_j + \frac{\varepsilon}{2\pi N} \sum_{i=1}^{N} \sin[2\pi(\theta^{(i)} - \theta^{(j)})] \qquad 1 \leqslant j \leqslant N. \tag{2}$$

In these models, $\theta^{(j)}$ and Ω_j are a phase and a natural frequency of the jth oscillator, respectively. (To be exact, Ω_j is a natural winding number in the model (1).) The latter is supposed to be distributed over the population with a density $f(\Omega)$. The parameter ε specifies the strength of interactions. For the model (2) it is known analytically that in the limit $N \to \infty$ an order parameter K equals zero for $\varepsilon \leqslant \varepsilon_c$ while it is positive for $\varepsilon > \varepsilon_c$ where K is the absolute value of $\lim_{t\to\infty} N^{-1} \sum_{j=1}^{N} \exp 2\pi \mathrm{i}' \theta^{(j)} (\mathrm{i}' \equiv (-1)^{1/2})$ and the threshold ε_c coincides with the onset of macroscopic entrainment [2, 6–8]. For example, if

$$f(\Omega) = (\gamma/\pi)[(\Omega - \tilde{\Omega})^2 + \gamma^2]^{-1} \tag{3}$$

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ε_c equals $4\pi\gamma$ at which a ME sets in with the common frequency $\tilde{\Omega}$. A similar behaviour was found numerically in the discrete-time model (1) as well [3, 4]. (We remark that discrete-time models such as (1) were introduced to investigate the dynamics of populations of quasiperiodic oscillators [3], whereas continuous-time models, including the one in (2), are for populations of limit-cycle oscillators (see [1, 2] for the expected scientific significance of the latter models). As for the former type of models, we refer the reader to an earlier paper [3] where detailed arguments are given on why investigating them is important and interesting.)

A most remarkable discovery in the earlier work [3], however, is the anomalous enhancement of persistent fluctuations of the order parameter in the neighbourhood of ε_c. In order to explain this, let us recall details of the previous computations. The natural winding numbers were chosen as

$$\Omega_j = \tilde{\Omega} + \gamma \tan[(j\pi/N) - (N+1)\pi/2N] \qquad 1 \leqslant j \leqslant N \tag{4}$$

whose distribution is given by (3) in the limit $N \to \infty$. For convenience, computations were carried out for $\psi_n^{(j)} \equiv \theta_n^{(j)} - n\tilde{\Omega}$ $(1 \leqslant j \leqslant N)$ with a particular initial condition: $\psi_0^{(j)} = 0$ $(1 \leqslant j \leqslant N)$. Evolution equations of $\psi^{(j)}$ may then be simplified for even N as follows:

$$\psi_{n+1}^{(j)} = \psi_n^{(j)} + \Delta_j - (\varepsilon X_n/2\pi)\sin 2\pi\psi_n^{(j)} \qquad 1 \leqslant j \leqslant N \tag{5}$$

where $\Delta_j \equiv \Omega_j - \tilde{\Omega}$ and $X_n \equiv N^{-1}\Sigma_{j=1}^N \cos 2\pi\psi_n^{(j)}$, because $Y_n \equiv N^{-1}\Sigma_{j=1}^N \sin 2\pi\psi_n^{(j)} = 0$ for all $n \geqslant 0$. Results based on (5) with $N = 100$ and $\gamma = 10^{-3}$ indicated the presence of persistent fluctuations in X_n. Therefore, two quantities were computed, i.e. $\langle X\rangle$ and $\sigma \equiv (\langle X^2\rangle - \langle X\rangle^2)^{1/2}$, where the brackets $\langle\ \rangle$ stand for a long time average. As the parameter ε was varied, $\langle X\rangle$, which may be identified with K, was found to behave qualitatively in the same way that K does in model (2). On the other hand, σ behaved as if it were divergent at the threshold. This suggests divergence of $\tilde{\sigma}$ at $\varepsilon = \varepsilon_c$, where $\tilde{\sigma}$ is defined by

$$\tilde{\sigma} = \lim_{N\to\infty} \sqrt{N}\sigma. \tag{6}$$

(Note that by definition, $\sigma \leqslant 1 < \infty$.) The factor $\sqrt{N}$ comes from a naive expectation based on the central limit theorem. A numerical check on this will be done later. Although comparison with the numerical results was not made, a preliminary prediction was presented as follows:

$$\tilde{\sigma} \propto |\varepsilon - \varepsilon_c|^{-\alpha_\pm} \qquad \varepsilon \gtrless \varepsilon_c \tag{7}$$

near the threshold, where $\alpha_+ = \alpha_- = \frac{1}{2}$ [3].

Quite recently, however, Nishikawa and Kuramoto [9] argued analytically that $\tilde{\sigma}$ should remain finite even at the threshold in the case of model (2). Since the width of the distribution of Ω_j, γ in our computations happened to be fairly small, they claimed that the numerical results were essentially for model (2) and hence that the divergence of $\tilde{\sigma}$ such as (7) would not take place. The main purpose of the present letter is to numerically check in detail the critical behaviour of $\tilde{\sigma}$ in model (1) for larger N than taken previously. It was found that $\tilde{\sigma}$ still tends to diverge as (7), but with the exponents near $\frac{1}{8}$. We also attempt to numerically establish a scaling behaviour of K. Finite-size scaling forms of K and σ are also proposed on the basis of a heuristic argument and are verified numerically. In the following, the details of computations are the same as described above, except for N. (In particular, γ is fixed at 10^{-3}.) The iteration number for computing K and σ is typically $2^{19} = 524\,288$ throughout this letter for which convergence of both quantities turned out to be fairly good in general.

Let us first focus on the scaling behaviour of $K = \langle X \rangle$ at the onset of ME which is expected theoretically as follows [2-4, 7-8]:

$$K \propto (\varepsilon - \varepsilon_c)^{1/2} \tag{8}$$

where $\varepsilon_c = 2(e^{2\pi\gamma} - 1)$ [3, 4]. We wish to establish the exponent $\frac{1}{2}$ in (8) since in previous numerical investigations, including those for model (2) [2, 3, 7, 8], the behaviour of K was studied only in a rather global range of a control parameter than locally enough to confirm (8). Our result of a K against ε plot near the threshold is displayed in figure 1(a) for $N = 100$, 400, 800 and 1600.

It is evident that the data for different N almost equal one another far beyond the threshold, revealing good convergence of K for increasing N. On the other hand, as the threshold is approached, scattering of the data becomes more and more conspicuous. This remarkable finite-size effect at the onset of ME will be discussed in detail later.

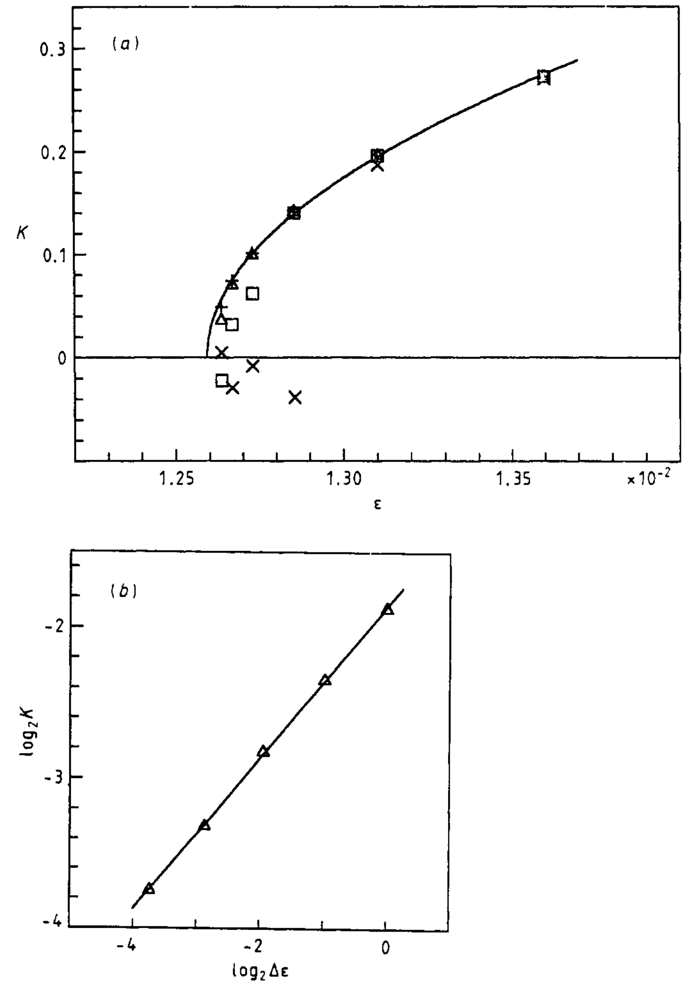

Figure 1. (a) K against ε near ε_c for $N = 100$ (×), 400 (□), 800 (△) and 1600(+). (b) The power law behaviour of K against $\Delta\varepsilon \equiv (\varepsilon - \varepsilon_c)/(\hat{\varepsilon} - \varepsilon_c)$, where $\varepsilon_c = 0.012\,593$ and $\hat{\varepsilon} = 0.0136$. The data are for $N = 1600$. The straight line is drawn by the least squares method and its slope is 0.502. The curve in (a) corresponds to the straight line in (b).

Concerning the location of the threshold, we made an extrapolation based on (8) using two of the data points for $N = 1600$ to obtain $\varepsilon_c = 0.012\,593$ which is in reasonable agreement with $0.012\,6059\ldots$ due to the formula for ε_c. (For $\gamma = 10^{-3}$, ε_c of model (2) is $0.012\,5663\ldots$ which is slightly smaller than that of model (1).) Figure 1(*b*) shows a log-log plot of the data for $N = 1600$ whose slope is 0.502 ± 0.003, being in excellent agreement with the theory. This result seems to suggest that, in the regime studied above, $N = 1600$ is large enough to extract critical scaling properties in the limit $N \to \infty$.

We now proceed to see the behaviour of fluctuations. Let us first confirm that σ is $O(N^{-1/2})$. Figure 2 shows $\log_2 \sigma$ against $\log_2(N/100)$ plots for $\varepsilon = 0.0116 (< \varepsilon_c)$ and $\varepsilon = 0.0136 (> \varepsilon_c)$. The power law behaviour of σ is clear. The slope of the data is -0.516 ± 0.002 for the former value of ε and -0.521 ± 0.002 for the latter. Both of these slopes agree reasonably well witth $\frac{1}{2}$ though they are somewhat larger. It has also been confirmed that the distribution of X_n is fitted well by a Gaussian law with mean $\langle X \rangle$ and variance σ^2 in accord with the central limit theorem [4]. We thus arrive at a point where the main subject of this letter is touched upon. A plot of $\sqrt{N}\sigma$ against ε around ε_c is presented in figure 3 for $N = 400$, 800 and 1600. This plot apparently suggests a divergence of σ at $\varepsilon = \varepsilon_c$. As a detailed check, log-log plots of $\sqrt{N}\sigma$ against $|\varepsilon - \varepsilon_c|$ for $N = 1600$ are displayed in figure 4 where $\varepsilon_c = 0.012\,593$ is used. (Of the data in figure 3, some whose ε is too close to ε_c are not included since the finite-size effect is expected to be serious for them.) The plots provide evidence for the power law divergence of $\tilde{\sigma}$ as expressed in (7) but the exponents are found to be drastically different from $\frac{1}{2}$, as follows: $\alpha_- = 0.126 \pm 0.009$ and $\alpha_+ = 0.123 \pm 0.003$, both of which are fairly close to $\frac{1}{8}$. In the same way, the exponents for $N = 800$ were found to be $\alpha_- = 0.120 \pm 0.004$ and $\alpha_+ = 0.132 \pm 0.005$ ($\varepsilon_c = 0.012\,606$ obtained by extrapolation was used). These results indicate that the increase in N from 800 to 1600 yields no serious change in the exponents. Moreover, the results (especially for $N = 1600$) appear to suggest $\alpha_+ = \alpha_-$. Thus a summary may be given in the following way: our results suggest that $\tilde{\sigma}$ behaves as (7) on both sides of ε_c with a common exponent near $\frac{1}{8}$, at

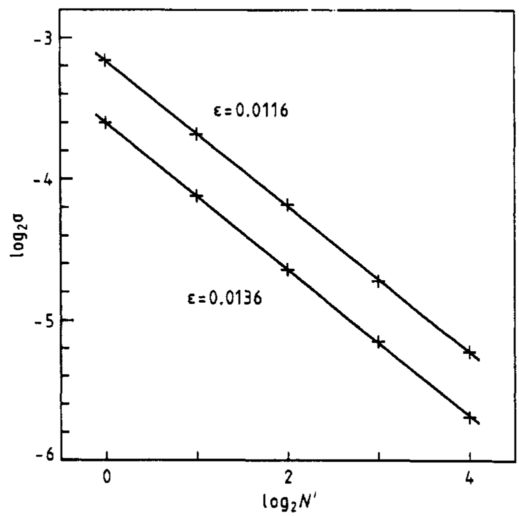

Figure 2. The power law behaviour of σ against $N (N' \equiv N/100)$ for $\varepsilon = 0.0116$ ($< \varepsilon_c$) and for $\varepsilon = 0.0136$ ($> \varepsilon_c$). For their slopes see the text.

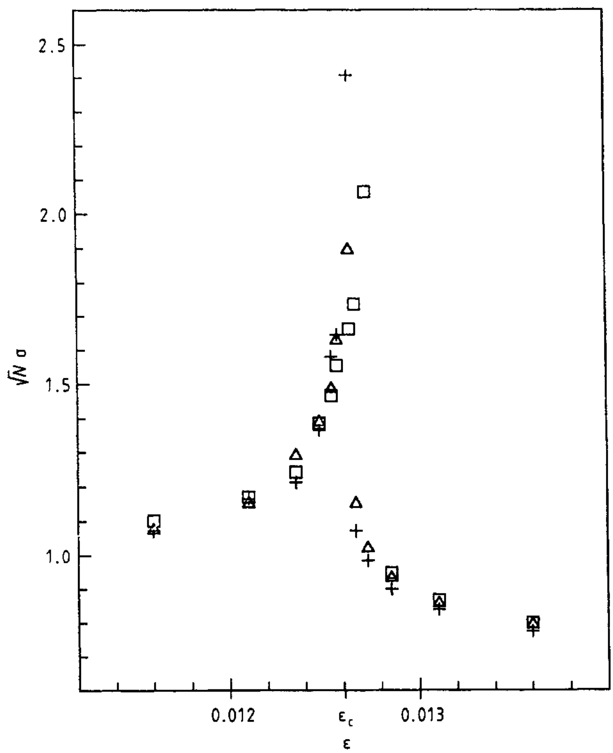

Figure. 3. The critical behaviour of σ multiplied by $\sqrt{N}$. The data are for $N = 400$ (□), 800 (△) and 1600 (+). The threshold marked under the abscissa is the theoretical one: 0.012 6059 . . . .

least in the investigated regime of ε. (We remark that the minimum of $|(\varepsilon - \varepsilon_c)/\varepsilon_c|$ for the data used in figure 4 is 0.004 for $\varepsilon < \varepsilon_c$ and 0.006 for $\varepsilon > \varepsilon_c$.)

Now let us look back at the behaviour of K displayed in figure 1(a). The finite-size effect on the critical scaling of K may be understood as follows. It is easy to see that in the limit $\gamma \to 0$ (i.e. when model (1) is well approximated by model (2)) oscillators entrained with the 'frequency' $\tilde{\Omega}$ are those which satisfy

$$|\Delta_j| < \varepsilon K/(2\pi) \tag{9}$$

when $\varepsilon > \varepsilon_c$. Furthermore, note that in a finite-size system, the mean distance between neighbouring Ω_j is $O(N^{-1})$. Therefore, if K decreases to $O(N^{-1})$ as ε approaches ε_c from above, there will be no oscillator whose Ω_j meets the inequality (9) and for ε closer to ε_c then K should vanish. In other words, in finite-size systems, K may be expected to be discontinuous at the onset of ME. This picture seems consistent with the following numerical observation, i.e. the data in figure 1(a) which deviate from the scaling form are actually transient with the tendency of diminishing towards zero. It may be a delicate problem, however, because of the presence of fluctuations, whether K is rigorously discontinuous or not. The above argument may not be sufficient to establish the discontinuity of K, but what is most important is that it allows us to estimate ε, $\varepsilon_f(N)$, for which the finite-size effect begins to work:

$$\varepsilon_f(N) - \varepsilon_c = O(N^{-2}) \tag{10}$$

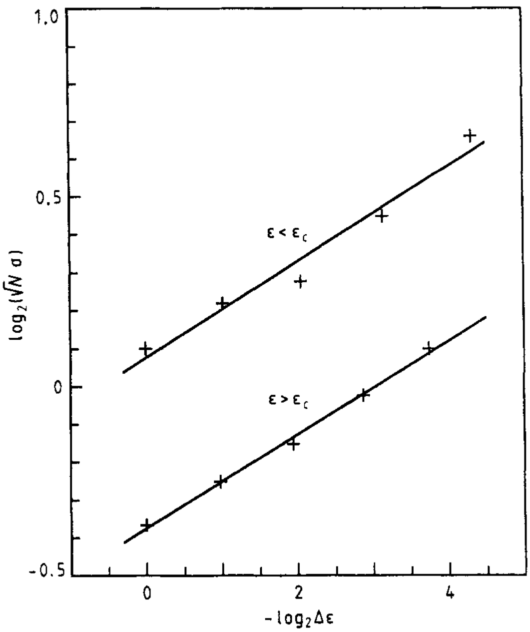

Figure 4. The power law behaviour of $\sqrt{N}\sigma$ against $\Delta\varepsilon$ which is $(\varepsilon_c-\varepsilon)/(\varepsilon_c-\bar{\varepsilon})$ for the upper $(\varepsilon<\varepsilon_c)$ and $(\varepsilon-\varepsilon_c)/(\hat{\varepsilon}-\varepsilon_c)$ for the lower $(\varepsilon>\varepsilon_c)$ plots where $\varepsilon_c=0.012\,593$, $\bar{\varepsilon}=0.0116$ and $\hat{\varepsilon}=0.0136$. Slopes of the data are given in the text.

where (8) has been used. Equation (10) suggests a finite-size scaling form of K as

$$K=N^{-1}\Phi(N^2\tilde{t}) \tag{11}$$

where $\tilde{t}\equiv\varepsilon-\varepsilon_c$ and $\Phi(x)$ is a function such that $\Phi(x)\propto x^{1/2}$ for $x\gg1$ (see [10] for the finite-size scaling analysis of equilibrium phase transitions). Numerical evidence for (11) is given in figure 5(a) where the theoretical value is used for ε_c. We may also expect a finite-size scaling of σ as

$$\sigma=N^{\beta}\Psi(N^2\tilde{t}) \tag{12}$$

where $\beta\equiv(4\alpha_+-1)/2$ and $\Psi(x)$ should satisfy $\Psi(x)\propto x^{-\alpha_+}$ when $x\gg1$. The best fit value of α_+ turned out to be 0.11 for the data of $N=800$, 1200 and 1600 and the result of that plot is shown in figure 5(b) supporting the scaling law of (12). (Since the above value of α_+ is affected by the data for smaller N, it would be less reliable than that found in figure 4 for $N=1600$.) This result provides us with further evidence for the divergence of $\tilde{\sigma}$ (for $\varepsilon>\varepsilon_c$). (We remark that in figure 5 the plots are presented for such a range of $N^2\tilde{t}$ that only the asymptotic behaviour of $\Phi(x)$ and $\Psi(x)$ for $x\gg1$ can be seen. This is because, owing to slow convergence, only some data were available in the remaining range closer to ε_c. It should also be mentioned that scalings of the data were found to be poor in the subcritical regime. This fact remains to be explained.)

Finally, note that the definition of σ used above is applicable only to a class of initial conditions for which $Y_n=0$ for all $n\geqslant0$. Generally, it should be defined as

$$\sigma=\langle|Z-\langle Z\rangle|^2\rangle^{1/2} \tag{13}$$

where $Z_n\equiv X_n+\mathrm{i}'Y_n$. Needless to say, it is important to confirm that σ is independent of the choice of an initial condition. This was done for $N=100$ and $\varepsilon=0.006$, 0.008,

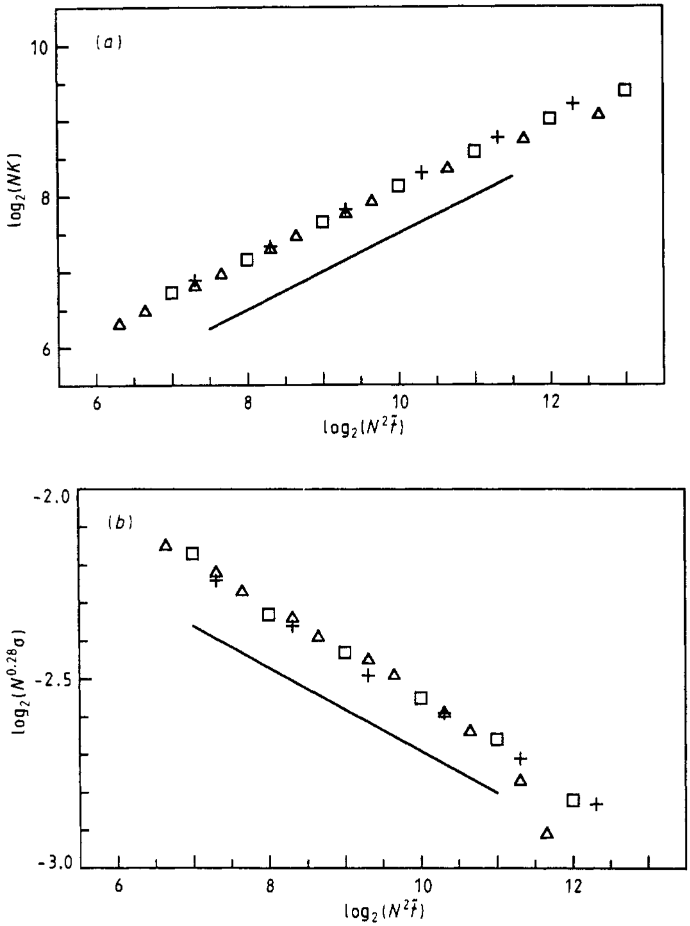

Figure 5. (a) Finite-size scaling of K (equation (11)) and of σ (equation (12) with $\alpha = 0.11$). In the abscissa, $\check{t}$ is $\varepsilon - \varepsilon_c$, where ε_c is the theoretical value. The data are for $N = 800$ (△), 1200 (□) and 1600(+). The straight line in (a) shows the slope $\frac{1}{2}$ and in (b) the slope -0.11.

0.015, 0.018 with a random initial condition and another one: $\psi_0^{(j)} = 0.2$ $(1 \leqslant j \leqslant N/2)$, 0.7 $(N/2 < j \leqslant N)$. σ thus computed using (13), as well as $K = |\langle Z \rangle|$, agreed with the results for the case of $\psi_0^{(j)} = 0$ $(1 \leqslant j \leqslant N)$ obtained using (5).

In summary we have found numerical results suggesting that $\tilde{\sigma}$, a measure of fluctuations related to the order parameter K, diverges at the threshold of ME as (7) with the same exponents close to $\frac{1}{8}$ on both sides of the threshold, being in contradiction with the recent claim by Nishikawa and Kuramoto [9] as well as the author's preliminary prediction [3]. The critical behaviour of $\tilde{\sigma}$ is very interesting because it is not of the mean-field type despite the fact that model (1) possesses a clear mean-field character. (In fact, such a character is reflected by the behaviour of K (8).) Of course, we still have to be careful since a crossover may happen even in the infinite system in a deeper critical regime than investigated above. (Note that such a crossover should obviously exist in any finite system since $\sqrt{N}\sigma$ is bounded by $\sqrt{N}$.) Further investigations would be necessary to check this point and, in addition, to see how universal the exponents are. We have also examined the scaling of K to find excellent agreement with the theory. Moreover, finite-size scaling forms of K and σ have been proposed on the basis of (10) and verified on the supercritical side. As to σ, such a scaling may be regarded as supporting the behaviour of $\tilde{\sigma}$ as (7) for $\varepsilon > \varepsilon_c$. To the author's knowledge, the present letter is the first to apply the finite-size scaling analysis [10] to the population dynamics of interacting oscillators. The utility of such an analysis should be noted.

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